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STRESS CONCENTRATION EFFECTS

IN MICROPOLAR ELASTICITY

by

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# STRESS CONCENTRATION EFFECTS IN MICROPOLAR ELASTICITY

## Abstract

In the present note, consideration is given to the problem
of stress concentration around a circular hole in the theory of
micropolar elasticity. The results obtained appear to contradict
the findings of Mindlin on the same problem based upon the
linear theory of couple stress elasticity.

1. Introduction

Eringen and Suhubi [1,2] have formulated and developed a continuum theory of micro-isotropic elastic solids, which takes into account the micro-motions and micro-rotation of the media. A limited case of this theory, called a determinate theory of couple stress elasticity, has also been deduced in [2]. Recently Eringen [3] has recapitulated this theory on a more sound footing, discussing the thermodynamics and uniqueness theorems, and has named it the linear theory of micropolar elasticity. In this theory, besides the macro deformation, the micro-rotation of the elements of the media, represented by a vector $\underset{\sim}{\Phi}$, is also taken into account. However, this vector $\underset{\sim}{\Phi}$, which is found to be responsible for developing a couple stress $m_{k\ell}$ in the media, is kinematically independent and is not related to the linear displacement, $\underset{\sim}{u}$. Physically the solids which are composed of dumbbell macro molecules, such as fibrous and coarse grain structure materials, are thought to be described by this theory. The constitutive equations for the stress tensor $t_{k\ell}$ and the couple stress $m_{k\ell}$ (in Cartesian tensor notation) for this class of materials are given as

$$t_{k\ell} = \lambda u_{r,r} \delta_{k\ell} + \mu(u_{k,\ell} + u_{\ell,k})$$
$$+ \kappa(u_{\ell,k} - \epsilon_{k\ell r} \Phi_r) \tag{1}$$

$$m_{k\ell} = \alpha \Phi_{r,r} \delta_{k\ell} + \beta \Phi_{k,\ell} + \gamma \Phi_{\ell,k} \tag{2}$$

where $\epsilon_{k\ell r}$ is the alternating tensor, $\lambda$ and $\mu$ are the classical

Lamé constants while $\kappa$ , $\alpha$ , $\beta$ and $\gamma$ are four additional constants which arise because of the consideration of the micro-structure of the media. Also the equations of balance of momentum and the balance of first stress moments take the form

$$t_{k\ell,k} + \rho\, f_\ell = \rho\, \ddot{u}_\ell \tag{3}$$

$$m_{rk,r} + \epsilon_{k\ell r}\, t_{\ell r} + \rho\, \ell_k = \rho\, \dot{\sigma}_k \tag{4}$$

Here, $f_\ell$ is the body force per unit mass, $\ell_k$ is the first body moment per unit mass, $\rho$ is the mass density and $\dot{\sigma}_k$ is the micro-rotation inertia of the elements. The quantity $\dot{\sigma}_k$ is expressed in terms of $\underset{\sim}{\Phi}$ by a relation of the form

$$\dot{\sigma}_k = J\, \ddot{\phi}_k \tag{5}$$

where $J$ is another elastic constant depending upon the material property.

Upon examining several uniqueness and energy theorems, Eringen [3] has also given the following thermodynamical restrictions upon the various constants, provided the internal energy has to be non-negative:

$$3\lambda + 2\mu + \kappa \geq 0 \ , \quad \mu \geq 0 \ , \quad \kappa \geq 0 \ ,$$

$$3\alpha + 2\gamma \geq 0 \ , \quad \gamma \geq \beta \geq -\gamma \ , \quad \gamma \geq 0 \tag{6}$$

The solution of the problem to be considered here has already been analyzed by Mindlin [4], but his derivations are based upon

a linearized theory of couple-stress elasticity developed by
Mindlin and Tiersten [5]*. This theory, although introduces
the consideration of couple stresses in continuum mechanics, is,
however, different from the theory of micropolar elasticity in
the following two significant aspects. In the first place, the
linearized theory of couple stress elasticity leaves the anti-
symmetric part of the stress tensor and also the symmetric part
of the couple stress tensor indeterminate, whereas the theory
of micropolar elasticity does not. In the former case, therefore,
it becomes difficult to have the clear and complete information
of the problems in which cross stresses play important roles.
Secondly, the linear theory of couple-stress elasticity defines
the rotation of the elements in terms of the displacement of
the elements, as in the classical sense, whereas the theory of
micropolar elasticity defines the micro-rotation kinematically
independently with the linear displacement. It, therefore, fol-
lows that in the former case the consideration of the rotation
depends solely upon the fact whether the element undergoes linear
displacement or not, whereas in the latter case the consideration
of the rotation of the elements is always possible whether or
not there is some linear displacement**. Finally, the theory of
couple-stress elasticity does not account for the inertia forces
which are equally important in dynamical problems.

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* For an account of this theory applicable to engineering problems
and also for the physical reality of the results based upon this
theory, we refer to the paper by Schijve [6].

** It has also been shown in [3] that linear theory of couple-stress
elasticity [5] turns out as a limited case of the linear theory of
micropolar elasticity under constrained motions.

Besides above facts, the theory of micropolar elasticity
has some physical precedent for it. In his study of dislocation
and moment stresses from the continuum approach, Kröner [7]
obtains an equation similar to Equation (2) for the torque
stress tensor connected with the lattice curvature. In our
present notation, if $m_{k\ell}$ is taken to represent the torque
stress, then $\Phi_{i,j}$ clearly represents the lattice curvature.
Also in agreement with Kröner, couple stresses in the micropolar
theory are, in fact, a consequence of the presence of dislocations,
i.e., lattice curvature, and therefore vanish whenever $\Phi$ vanishes.

With these considerations, we now proceed to discuss the
problem of stress concentration around a circular hole in an
infinite plate subject to axial tension. Our analysis reveals
that the effect of couple stresses is significantly small as
compared to the classical case; thus agreeing with the experi-
mental findings of Schijve [6], but contradicting the conclusions
of Mindlin [4] who claimed that couple stresses have considerable
influence. This striking difference with Mindlin's result, in
fact, arises because of a new ratio $(\frac{b}{c})$ , defined below, that
occurs in the present analysis but is not found in the case of
linear couple stress theory. However, since some of the steps
in our mathematical analysis are the same as those of Mindlin's
work, we omit the corresponding details here and refer the
reader to this paper for collateral study; the notations of
which would be used here as far as possible.

## 2. Stress Concentration Around A Circular Hole

The problem of stress concentration around a circular hole is the problem of two dimensional plane deformations in cylindrical polar coordinates. Hence on assuming the displacement and micro-rotation components of the form

$$u = u(r,\theta) \quad , \quad v = v(r,\theta) \quad , \quad \Phi_z = \Phi_z(r,\theta) \tag{7}$$

the equations of compatibility are given as in [8],

$$\frac{\partial \bar{e}_{\theta r}}{\partial r} + \frac{\bar{e}_{\theta r} + \bar{e}_{r\theta}}{r} - \frac{1}{r}\frac{\partial \bar{e}_{rr}}{\partial \theta} - \frac{\partial \Phi_z}{\partial r} = 0 \ ,$$

$$\frac{\partial \bar{e}_{\theta\theta}}{\partial r} + (\frac{\bar{e}_{\theta\theta} - \bar{e}_{rr}}{r}) - \frac{1}{r}\frac{\partial \bar{e}_{r\theta}}{\partial \theta} - \frac{1}{r}\frac{\partial \Phi_z}{\partial \theta} = 0 \ , \tag{8}$$

$$\frac{\partial m_{\theta z}}{\partial r} + \frac{m_{\theta z}}{r} = \frac{1}{r}\frac{\partial m_{rz}}{\partial \theta}$$

where

$$\bar{e}_{k\ell} = u_{\ell,k} - \epsilon_{k\ell r}\Phi_r \ , \tag{9}$$

On introducing the stress functions $\varphi(r,\theta)$ and $\psi(r,\theta)$ consistent to Equations (3) and (4) (cf. Mindlin)

$$t_{rr} = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} \quad ,$$

$$t_{\theta\theta} = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} \quad ,$$

$$t_{r\theta} = -\frac{1}{r}\frac{\partial^2 \varphi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta} - \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \quad , \tag{10}$$

$$t_{\theta r} = -\frac{1}{r}\frac{\partial^2 \varphi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta} + \frac{\partial^2 \psi}{\partial r^2} \quad ,$$

$$m_{rz} = \frac{\partial \psi}{\partial r} \quad , \quad m_{\theta z} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad .$$

we find that compatibility equations, after some calculations, become

$$\frac{\partial}{\partial r}(\psi - c^2 \nabla^2 \psi) = -2(1-v') b^2 \frac{1}{r}\frac{\partial}{\partial \theta} \nabla^2 \varphi$$

$$\tag{11}$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(\psi - c^2 \nabla^2 \psi) = +2(1-v') b^2 \frac{\partial}{\partial r} \nabla^2 \varphi$$

where

$$c^2 = \frac{\gamma(\mu + \kappa)}{(2\mu+\kappa)\kappa} \quad , \qquad b^2 = \frac{\gamma}{2(2\mu+\kappa)}$$

$$v' = \frac{\lambda}{(2\mu+2\lambda+\kappa)} \quad , \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{12}$$

From Equations (11) we can easily arrive at the equations

$$\nabla^4 \varphi = 0$$

$$\nabla^2(\psi - c^2 \nabla^2 \psi) = 0 \tag{13}$$

which are now uncoupled between $\varphi$ and $\psi$. Except with the replacement of $c^2$ for $\ell^2$, where now $c^2$ is expressed in terms of the elastic constants having the dimensions of length square, Equations (13) are identical with those obtained by Mindlin. The compatibility Equations (11) are, however, somewhat different from those of Mindlin.

Mindlin takes suitable forms for $\varphi$ and $\psi$ and then, by applying appropriate boundary conditions, determines the expressions for stress and couple stress components. The particular result of interest, he calculates, is the expression for $t_{\theta\theta}$ (max), which occurs at $r = a$ and $\theta = \pm \frac{\pi}{2}$, and turns out to be (cf. Mindlin's Equation (40))

$$t_{\theta\theta} \text{ (max)} = p \frac{3 + F}{1 + F} \tag{14}$$

where $p$ is the magnitude of the uniform tension applied in the x-direction and $F$ is given by (cf. Mindlin's Eq. (39))

$$F = 8(1 - \nu) \left[ 4 + \frac{a^2}{\ell^2} + \frac{2a}{\ell} \frac{K_o(a/\ell)}{K_1(a/\ell)} \right]^{-1} \tag{15}$$

where $a$ is the radius of the circular hole. It is thus clear from (14) that the influence of couple stresses is mainly affected due to the magnitude of $F$, given by (15), which in turn depends upon $\nu$, $a$ and $\ell$. Of these three the first two could be known easily while the third factor $\ell$ arises due to the presence of couple stresses.

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Mindlin, in giving the physical background of $\ell$ states, "In perfect crystals and amorphous materials like glass ... dimensions of holes, fillets, notches or cracks in them may approach $\ell$ for a variety of materials." That this reasoning is not sound could be argued from the fact that in notched specimen considerations, the apparent rise of the yield stress is, in fact, because of the difficulty of indicating the beginning of the plastic deformation and that in such cases yielding is a localized phenomenon.

In our present calculations it is found that $F'$, the corresponding value of $F$ to that of Mindlin's case, is given as

$$F' = \frac{8(1 - \nu')\ b^2/c^2}{[4 + \dfrac{a^2}{c^2} + \dfrac{2a}{c}\ \dfrac{K_0(a/c)}{K_1(a/c)}\ ]} \tag{16}$$

A careful comparison of (15) and (16) indicates that $F'$ can go over to $F$ only if

$$\frac{b^2}{c^2} = 1\quad,\quad c = \ell\quad,\quad \nu' = \nu \tag{17}$$

Whether or not the second and third conditions of (17) hold, the first condition itself, with the help of Equation (12), reduces

$$\frac{1}{2}\left(\frac{\kappa}{\mu + \kappa}\right) = 1\qquad \text{i.e.,}\qquad \kappa = -2\mu \tag{18}$$

which not only contradicts the thermodynamical restrictions given by (6) but also is physically unrealistic in the sense that the magnitude of the constant $\kappa$ cannot be as great as to be equal to the twice of the shear modulus. In view of this fact it can,

therefore, be concluded that expression for $F$ , given by (15), is not acceptable.

On using $F'$ , in place of $F$ , in Equation (14), calculations have been made again for different reasonable values of $\kappa$ and $\gamma$ . The physical ground for the particular values of $\kappa$ used here is based on the fact that in polycrystalline materials the magnitude of shear-modulus $((2\mu + \kappa)$ in our calculations) is increased by nearly five percent of its usual value, as has been observed experimentally in many cases, for example, by Bradfield and Pursey [9]. The values of $\gamma$ are, however, taken from a corresponding calcu-lation of Kröner in which the ratio of the torque stress to the lattice curvature has been obtained. In the present case it is found that stress concentration factor not only depends upon $(\frac{a}{c})$ $(\frac{a}{l}$ that of Mindlin's case) and the Poisson's ratio but also is controlled by a new ratio $(\frac{b}{c})$ . With the above choice, for values of $\kappa$ and $\gamma$ , the values of $(\frac{b}{c})$ and $c$ are found to be significantly small. It then follows that the value of $\frac{a}{c}$ would be considerably high since the radius of the hole cannot be below the order of 1 mm. Figures 2 to 5 show the plot of stress concen-tration factor against $(\frac{a}{c})$ for two different values of $\frac{b}{c}$ . A clear comparison between Eringen's theory and Mindlin's theory has been displayed in the graphs. It can be easily seen that, in the present case, the stress concentration factor ranges from 2.97 to 2.985 (even for an extreme case when $\frac{a}{c} = 3$ , $\frac{b}{c} = 0.20$), very near to the classical value, whereas Mindlin's theory predicts a range from 2.4 to 2.6.

As a final remark it may also be worthwhile to point out here that the conclusions drawn by Cohen [10] and Muki and Sternberg [11],

on the basis of the linear theory of couple stress elasticity, would be considerably affected in a likewise manner as in the case of the present analysis. We do not, however, wish to pursue this analysis in the present note.
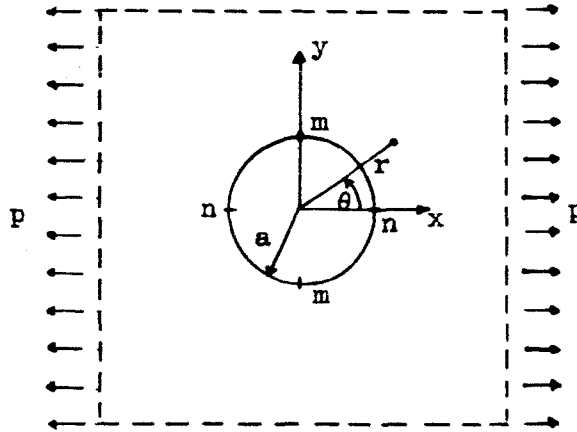

### Adknowledgement

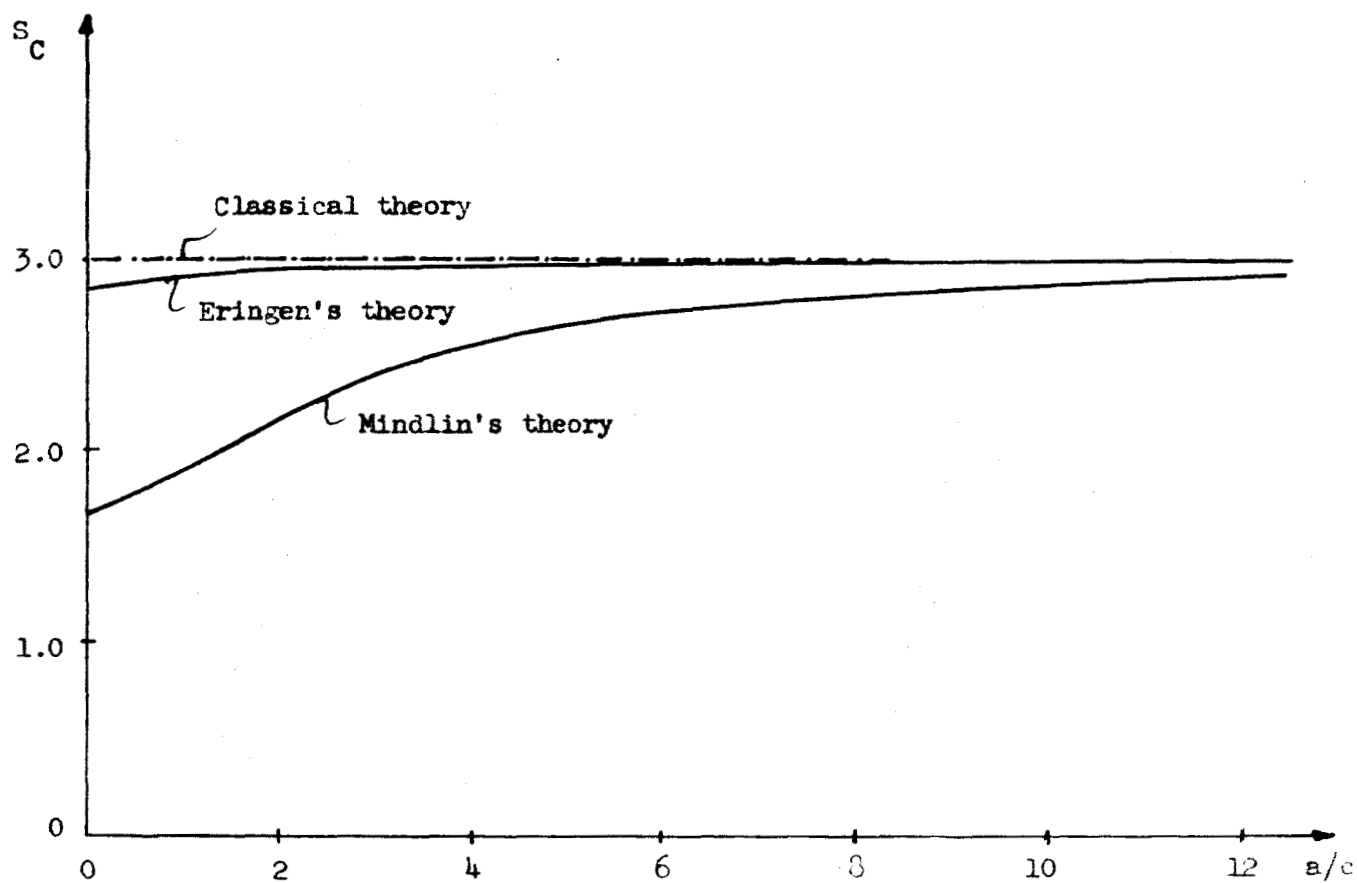Fig. 1  Circular hole in an infinite plate
subject to axial tension.

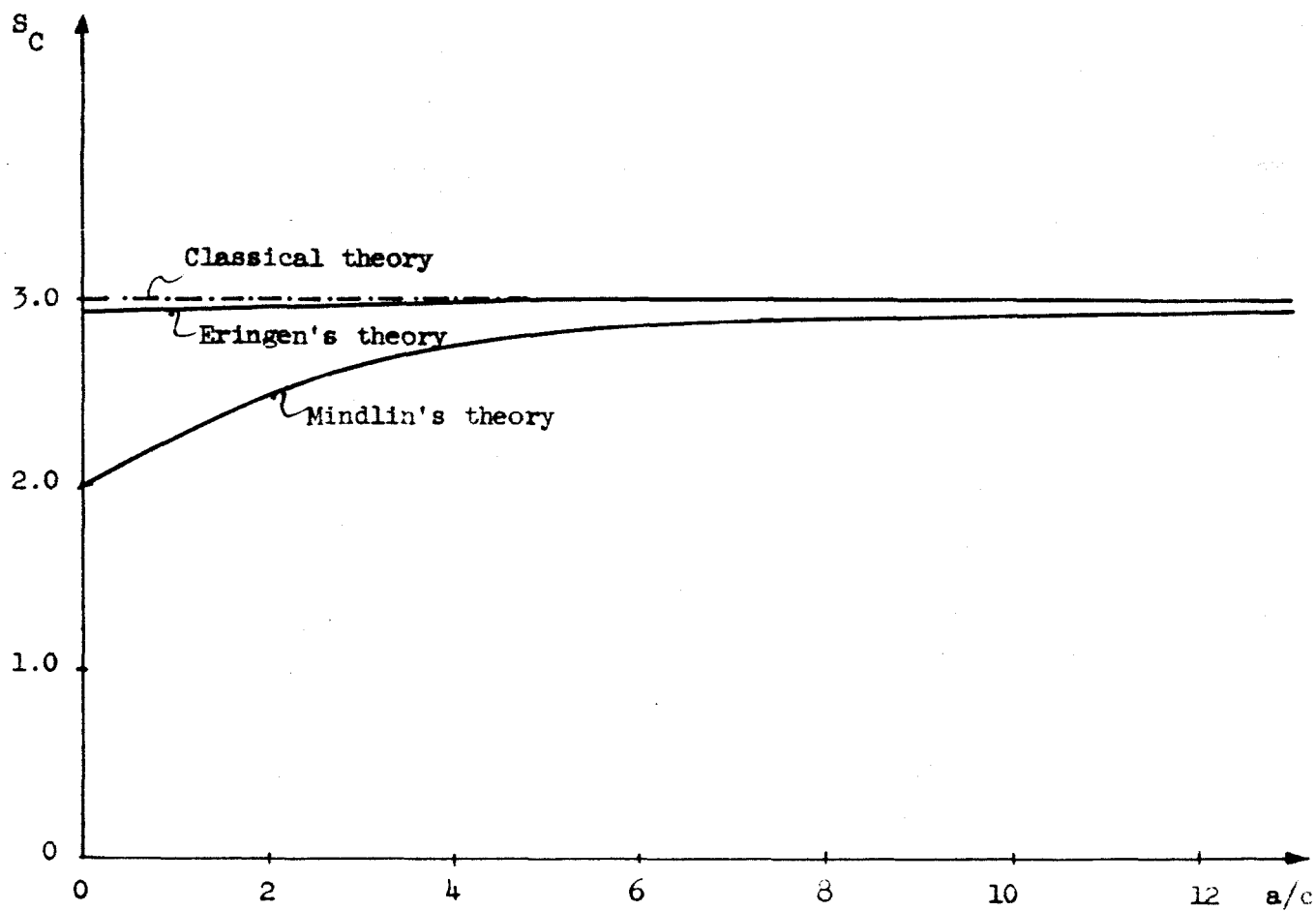Fig. 2  Stress Concentration Factors for $\frac{b}{c} = 0.20$ , $\nu = 0$ .

Fig. 3  Stress Concentration Factors for $\frac{b}{c} = 0.20$ , $\nu = 0.5$.
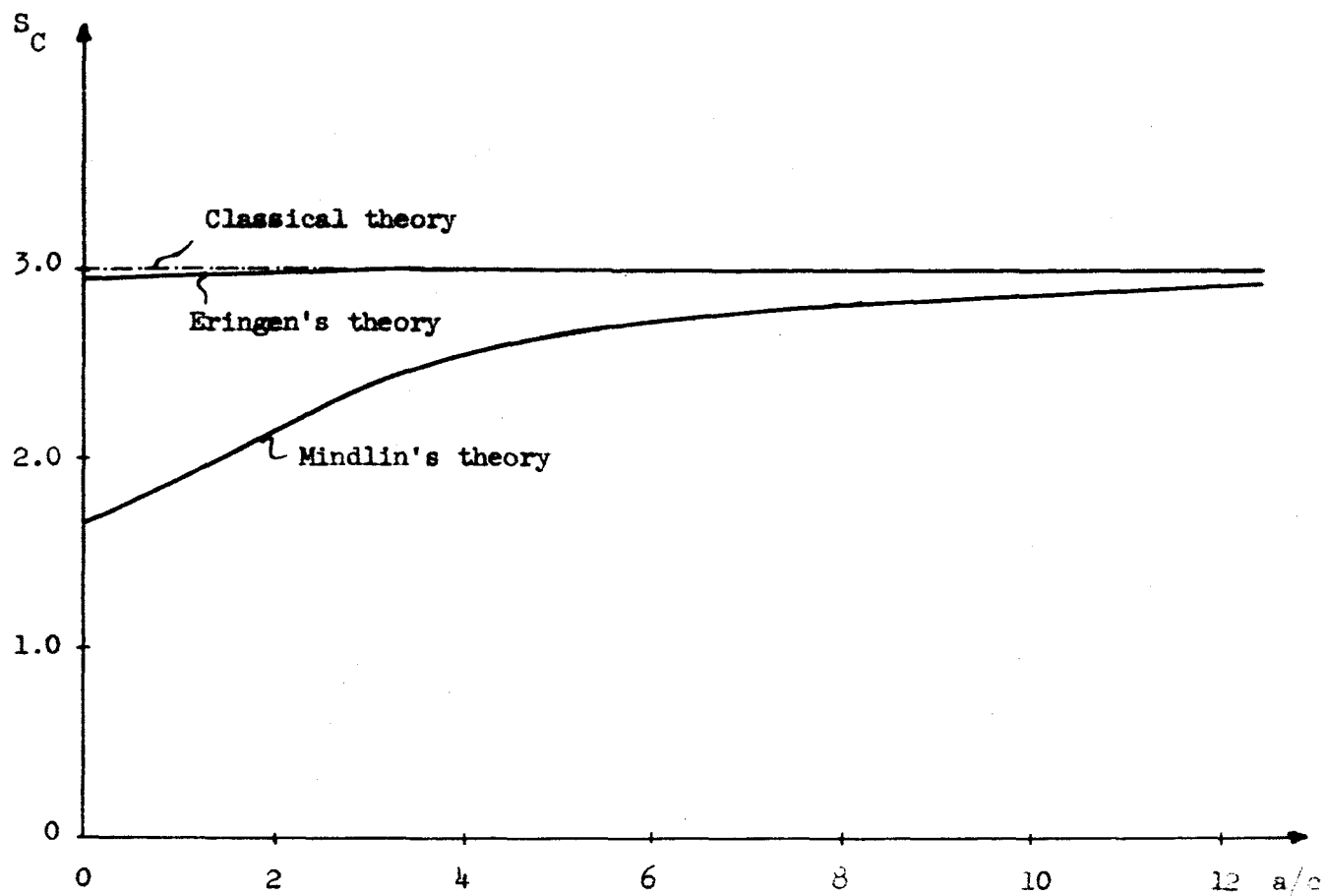
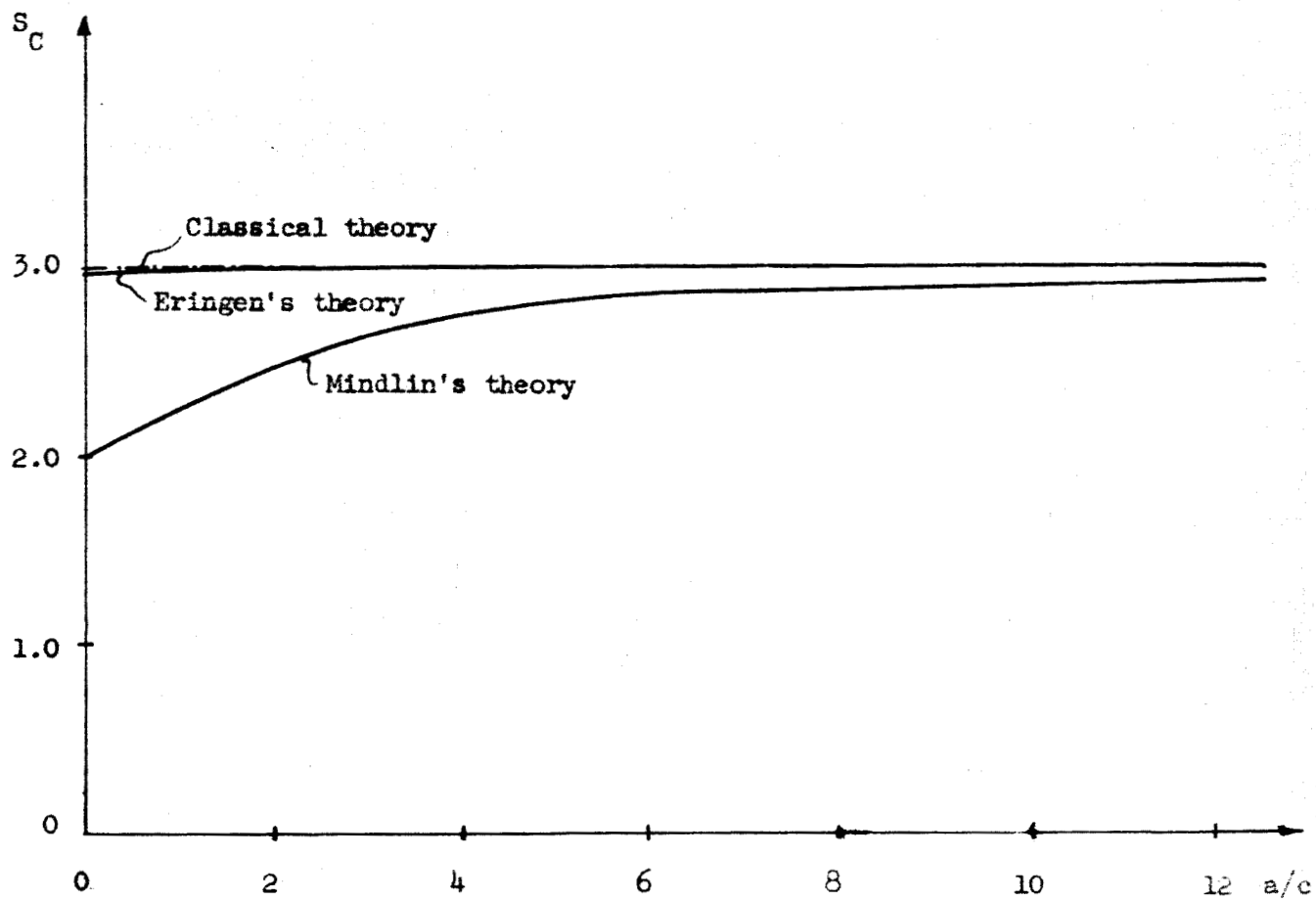Fig. 4 Stress Concentration Factors for $\frac{b}{c} = 0.10$ , $\nu = 0$ .

Fig. 5  Stress Concentration Factors for $\frac{b}{c} = 0.10$ , $\nu = 0.5$.

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